

# vDVZ $\Leftarrow$ Fierz-Pauli

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## Abstract

Perturbative vDVZ discontinuity as an artifact of the Fierz-Pauli mass term becomes evident in the massless limit of gravitational Higgs mechanism.

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# 1 Introduction and Summary

A general Lorentz invariant mass term for the graviton  $h_{MN}$  in the linearized approximation is of the form

$$-\frac{m^2}{4} [h_{MN}h^{MN} - \beta(h_M^M)^2] , \quad (1)$$

where  $\beta$  is a dimensionless parameter. Perturbatively, for  $\beta \neq 1$  the trace component  $h_M^M$  is a propagating ghost, while it decouples in the Minkowski background for the Fierz-Pauli mass term with  $\beta = 1$  [1]. Gravitational Higgs mechanism [2, 3] provides a non-perturbative and fully covariant definition of massive gravity. Non-perturbatively, even for  $\beta \neq 1$ , the Hamiltonian is bounded from below and the perturbative ghost is an artifact of linearization [4].<sup>3</sup>

For  $\beta = 1$  perturbative radially symmetric asymptotic solutions are singular in the  $m \rightarrow 0$  limit: we have the van Dam-Veltman-Zakharov (vDVZ) discontinuity [6, 7] and we must consider non-perturbative solutions [8]. In this note, following the method of [9], we argue that for  $\beta \neq 1$  perturbative solutions have a smooth massless limit, hence no vDVZ discontinuity. Simply put, the perturbative vDVZ discontinuity is an artifact of the Fierz-Pauli mass term. This becomes particularly evident in the language of constrained gravity, which is the massless limit of gravitational Higgs mechanism [9].

## 2 Gravitational Higgs Mechanism

In this section we very briefly review gravitational Higgs mechanism and discuss its massless limit. We have gravity in  $D$  dimensions coupled to scalar fields  $\phi^A$ ,  $A = 0, \dots, D-1$ . Coordinate-dependent scalar VEVs break diffeomorphisms spontaneously. Because diffeomorphisms are broken spontaneously, the  $D$  scalars  $\phi^A$  can be gauge-fixed to their background values, which leaves massive gravity. The resulting action for gravity is given by

$$S = M_P^{D-2} \int d^D x \sqrt{-G} [R - \mu^2 V] , \quad (2)$$

where  $\mu$  has the dimension of mass, and  $V$  is a dimensionless “potential” that makes bulk gravity massive and *a priori* is a generic function constructed from the metric  $G_{MN}$ , antisymmetric tensor density  $\epsilon_{M_1 \dots M_D}$ , and the background metric  $E_{MN}$ . For our purposes here it will suffice to consider potentials of the form  $V = V(X)$ , where  $X \equiv G^{MN} E_{MN}$ . The equations of motion read

$$R_{MN} = \mu^2 \left[ V'(X) E_{MN} + \frac{V(X) - X V'(X)}{D-2} G_{MN} \right] , \quad (3)$$

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<sup>3</sup>The full non-perturbative Hamiltonian for the model of [3], which has  $\beta = 1/2$ , in the gravitational Higgs mechanism framework was constructed in [5] and is expressly positive-definite. Non-perturbative unitarity for general  $\beta$  was argued in [4].

with the Bianchi identity

$$\partial_M \left[ \sqrt{-G} V'(X) G^{MN} E_{NS} \right] - \frac{1}{2} \sqrt{-G} V'(X) G^{MN} \partial_S E_{MN} = 0 , \quad (4)$$

which is equivalent to the gauge-fixed equations of motion for the scalars. In (2) we have deliberately omitted any source terms. In this note we will focus on the cases with Ricci-flat background metric  $E_{MN}$ , which implies that  $V(D) = 2V'(D)$ .

In the linearized approximation the r.h.s. of (3) corresponds to the graviton mass term (1) with

$$m^2 \equiv 2\mu^2 V'(D) , \quad (5)$$

$$\beta \equiv \frac{1}{2} - \frac{V''(D)}{V'(D)} . \quad (6)$$

We have  $\beta = 1$  for potentials  $V$  with  $V'(D) = -2V''(D)$ . For a linear potential  $V(X) = a + X$ , we have the model of [3] with  $\beta = 1/2$ . *E.g.*, for quadratic potentials  $V = a + X + \lambda X^2$  with  $\lambda \neq 0$  we can have other values of  $\beta$ , including  $\beta = 1$  for  $\lambda = -1/2(D+2)$ .

## 2.1 Constrained Gravity as the Massless Limit

The massless limit  $m \rightarrow 0$  corresponds to taking  $\mu \rightarrow 0$ . In this limit we obtain not Einstein-Hilbert gravity but *constrained* gravity [9]. This is because the Bianchi identity (4) survives in the massless limit. Here  $E_{MN}$  is the flat Minkowski metric  $\eta_{MN}$  if the coordinates  $x^M$  are Minkowski coordinates. However, in general the metric  $E_{MN}$  need not be the Minkowski metric. For instance, in spherical coordinates we have

$$E_{MN} dx^M dx^N = -dt^2 + dr^2 + r^2 \gamma_{ab} dx^a dx^b , \quad (7)$$

where  $\gamma_{ab}$  is a metric on the unit sphere  $S^{d-1}$ ,  $d \equiv D - 1$ .

The fact that we obtain constrained gravity in the massless limit is important. If we take, say, a spherically symmetric solution in massive gravity and consider the massless limit, it need not coincide with the Schwarzschild solution of Einstein-Hilbert gravity. Instead, it should coincide with a spherically symmetric solution in constrained gravity. One way to construct solutions in constrained gravity is to start with known solutions in Einstein-Hilbert gravity and coordinate-transform them to satisfy the constraint [9] (this is similar to [10]).

## 3 Spherically Symmetric Solutions

For spherically symmetric solutions the metric reads ( $A, B, C$  are functions of  $r$  only):

$$ds^2 = -A^2 dt^2 + B^2 dr^2 + C^2 \gamma_{ab} dx^a dx^b \quad (8)$$

and we have

$$X = A^{-2} + B^{-2} + (D-2)r^2C^{-2} . \quad (9)$$

The non-vanishing components of  $R_{MN}$  are given by (prime denotes derivative w.r.t.  $r$ , not to be confused with derivative w.r.t.  $X$  as in  $V'(X)$ ):

$$R_{00} = A^2B^{-2} \left[ \frac{A''}{A} - \frac{A'B'}{AB} + (D-2)\frac{A'C'}{AC} \right] , \quad (10)$$

$$R_{rr} = - \left\{ \frac{A''}{A} - \frac{A'B'}{AB} + (D-2) \left[ \frac{C''}{C} - \frac{B'C'}{BC} \right] \right\} , \quad (11)$$

$$R_{ab} = -\gamma_{ab}R_* , \quad (12)$$

$$R_* \equiv C^2B^{-2} \left\{ \frac{C''}{C} + (D-3) \left( \frac{C'}{C} \right)^2 + \frac{C'}{C} \left[ \frac{A'}{A} - \frac{B'}{B} \right] \right\} - (D-3) . \quad (13)$$

The Bianchi identity (4) reduces to a single equation:

$$\partial_r [AB^{-1}C^{D-2}Q] - (D-2)rABC^{D-4}Q = 0 , \quad (14)$$

where  $Q \equiv V'(X)$ . We will focus on  $D = 4$  for the remainder of this paper as the generalization to higher  $D$  is straightforward.

### 3.1 Four-dimensional Massless Solutions

Let us start by analyzing the above equations in the massless case ( $m^2 = 0$ ) in  $D = 4$ . We have the following equations

$$\frac{A''}{A} - \frac{A'B'}{AB} + 2\frac{A'C'}{AC} = 0 , \quad (15)$$

$$\frac{A''}{A} - \frac{A'B'}{AB} + 2 \left[ \frac{C''}{C} - \frac{B'C'}{BC} \right] = 0 , \quad (16)$$

$$\frac{C''}{C} + \left( \frac{C'}{C} \right)^2 + \frac{C'}{C} \left[ \frac{A'}{A} - \frac{B'}{B} \right] - B^2C^{-2} = 0 , \quad (17)$$

plus the constraint

$$\partial_r [AB^{-1}C^2Q] - 2rABQ = 0 . \quad (18)$$

If it were not for the constraint, we could simply take the Schwarzschild solution:

$$\overline{A} = \overline{B}^{-1} = \sqrt{1 - \frac{r_*}{r}} , \quad (19)$$

$$\overline{C} = r . \quad (20)$$

However, this solution does not satisfy the constraint.

There is a systematic way of finding solutions that satisfy the constraint by transforming known solutions that satisfy unconstrained Einstein's equations. Thus, we start from a known unconstrained solution given by  $\overline{A}, \overline{B}, \overline{C}$ , and transform the radial coordinate  $r \rightarrow f(r)$ . The resulting metric components are given by

$$A(r) = \overline{A}(f(r)) , \quad (21)$$

$$B(r) = \overline{B}(f(r))f'(r) , \quad (22)$$

$$C(r) = \overline{C}(f(r)) , \quad (23)$$

and they still satisfy the equations of motion. This is because the massless equations of motion possess full reparametrization invariance. The constraint then produces a second order differential equation for the function  $f(r)$ . Thus, starting with the Schwarzschild solution, we can obtain solutions satisfying the constraint by setting  $f(r) = C(r)$  in the above expressions, which gives a differential equation for  $C$ . We have:

$$A = \sqrt{1 - r_*/C} , \quad (24)$$

$$B = \frac{C'}{\sqrt{1 - r_*/C}} , \quad (25)$$

and the differential equation for  $C$  reads:

$$\partial_r [A^2 C^2 Q / C'] - 2r Q C' = 0 . \quad (26)$$

While (26) is highly non-linear, we can solve it in two regimes: near the horizon ( $C \rightarrow r_*$ ), and asymptotically ( $r \gg r_*$ ). Here we are interested in asymptotic solutions.

### 3.2 Perturbative Asymptotic Solutions

To find perturbative asymptotic solutions to (26), we set

$$C = r(1 + c) \quad (27)$$

and only keep terms linear in  $c$ . This is equivalent to assuming that  $c = \gamma(r_*/r) + \mathcal{O}(r_*/r)^2$ , keeping only the leading terms linear in  $(r_*/r)$  and solving for the coefficient  $\gamma$  by requiring that (26) is satisfied to this approximation. A little straightforward algebra gives

$$\gamma = \frac{1}{2} \frac{V'(4)}{V'(4) + 2V''(4)} = \frac{1}{4(1 - \beta)} , \quad (28)$$

where we have used (6). So, for  $\beta \neq 1$  we have a perturbative asymptotic solution in constrained gravity which is the massless limit of the corresponding perturbative asymptotic solution in massive gravity. This massless perturbative asymptotic solution is valid at distance scales  $r \gg r_1 \equiv \gamma r_* = r_*/4(1 - \beta)$ . As  $\beta \rightarrow 1$ , this distance scale  $r_1 \rightarrow \infty$ . This implies that we have the perturbative vDVZ discontinuity for  $\beta = 1$ , but not for  $\beta \neq 1$ .

### 3.2.1 Non-perturbative Asymptotic Solutions for $\beta = 1$

The above result shows that for  $\beta = 1$  the linearized approximation (27) breaks down and we must look for non-perturbative asymptotic solutions. We can find a solution via the following Ansatz:

$$C = r \left[ 1 + \alpha \left( \frac{r_*}{r} \right)^{\frac{1}{2}} + \eta \frac{r_*}{r} + \mathcal{O} \left( \frac{r_*}{r} \right)^{\frac{3}{2}} \right], \quad (29)$$

where  $\alpha$  and  $\eta$  are numerical coefficients to be determined. This solves (26) [9]:

$$A = 1 - \frac{r_*}{2r} + \mathcal{O} \left( \frac{r_*}{r} \right)^{\frac{3}{2}}, \quad (30)$$

$$B = 1 + \sqrt{\frac{8}{39}} \left( \frac{r_*}{r} \right)^{\frac{1}{2}} + \frac{r_*}{2r} + \mathcal{O} \left( \frac{r_*}{r} \right)^{\frac{3}{2}}, \quad (31)$$

$$C = r \left[ 1 + \sqrt{\frac{8}{39}} \left( \frac{r_*}{r} \right)^{\frac{1}{2}} + \eta \frac{r_*}{r} + \mathcal{O} \left( \frac{r_*}{r} \right)^{\frac{3}{2}} \right], \quad (32)$$

and  $\eta$  is an integration constant. This is because we started with the Schwarzschild solution and transformed it via  $r \rightarrow C(r)$ . The constraint (26) is a second order differential equation for  $C$ , whose solution contains two integration constants. However, because we drop subleading terms, the resulting equation effectively is only a first order equation for  $C$ , so we have one integration constant (and the second integration constant controls the subleading terms). It simply parameterizes the Schwarzschild solution in the transformed coordinate frame.

### 3.3 Four-dimensional Massive Solutions

We can derive the above result that there is no perturbative vDVZ discontinuity for  $\beta \neq 1$  by directly solving the massive equations of motion in the asymptotic regime. In four dimensions we have:

$$A^2 B^{-2} \left[ \frac{A''}{A} - \frac{A'B'}{AB} + 2 \frac{A'C'}{AC} \right] = \mu^2 \left\{ A^2 \frac{XV'(X) - V(X)}{2} - V'(X) \right\}, \quad (33)$$

$$\frac{A''}{A} - \frac{A'B'}{AB} + 2 \left[ \frac{C''}{C} - \frac{B'C'}{BC} \right] = \mu^2 \left\{ B^2 \frac{XV'(X) - V(X)}{2} - V'(X) \right\}, \quad (34)$$

$$C^2 B^{-2} \left\{ \frac{C''}{C} + \left( \frac{C'}{C} \right)^2 + \frac{C'}{C} \left[ \frac{A'}{A} - \frac{B'}{B} \right] \right\} - 1 = \mu^2 \left\{ C^2 \frac{XV'(X) - V(X)}{2} - r^2 V'(X) \right\}, \quad (35)$$

$$\partial_r [AB^{-1}C^2V'(X)] - 2rABV'(X) = 0, \quad (36)$$

where the last equations is the Bianchi identity.

### 3.3.1 Perturbative Asymptotic Solutions

Let

$$A = 1 + a , \quad (37)$$

$$B = 1 + b , \quad (38)$$

$$C = r(1 + c) . \quad (39)$$

Here we assume that  $a, b, c$  go to zero asymptotically, and in the equations of motion we keep only linear terms in  $a, b, c$ . As we will see in a moment, this approximation breaks down for small graviton mass when  $\beta = 1$ , hence the vDVZ discontinuity.

The above four equations of motion in the linearized approximation read:

$$a'' + \frac{2}{r}a' = m^2 [a - \nu z] , \quad (40)$$

$$a'' + 2c'' + \frac{4}{r}c' - \frac{2}{r}b' = m^2 [b - \nu z] , \quad (41)$$

$$c'' + \frac{4}{r}c' + \frac{1}{r}[a' - b'] - \frac{2}{r^2}[b - c] = m^2 [c - \nu z] , \quad (42)$$

$$a' - b' + 2c' - 2\nu z' - \frac{4}{r}[b - c] = 0 , \quad (43)$$

where  $z \equiv a + b + 2c$ ,  $\nu \equiv V''(4)/V'(4) = 1/2 - \beta$  (see (6)), and  $m^2 = 2\mu^2 V'(4)$  (see (5)). From the above four equations we have the following equation for  $z$ :

$$2(1 - \beta) \left[ z'' + \frac{2}{r}z' \right] = (4\beta - 1)m^2 z . \quad (44)$$

For  $\beta = 1$  we therefore have  $z = 0$ , which is simply a manifestation of the fact that perturbatively the trace of the graviton is not a propagating degree of freedom, and

$$a = \frac{\zeta}{r} e^{-mr} , \quad (45)$$

$$b = \frac{\zeta}{m^2 r^3} [1 + mr] e^{-mr} , \quad (46)$$

$$c = -\frac{\zeta}{2r} e^{-mr} - \frac{\zeta}{2m^2 r^3} [1 + mr] e^{-mr} , \quad (47)$$

where  $\zeta$  is an integration constant. The only way to have a smooth massless limit would be to take  $\mu \rightarrow 0$  and  $\zeta \rightarrow 0$  with  $|\zeta|/m^2 \equiv r_2^3$  fixed. In this case in the massless limit we would have  $a = 0$  and  $b = -2c = r_2^3/r^3$ . However, the corresponding metric is equivalent to a coordinate-transformed flat metric (in spherical coordinates). So, for  $\beta = 1$  we have the perturbative vDVZ discontinuity. However, this discontinuity is an artifact of the perturbative approximation, which breaks down at  $r \sim r_2$ . Note that  $|\zeta|$  is expected to be of order of the Schwarzschild radius  $r_*$ , so we have  $r_2 \sim (r_*/m^2)^{1/3}$ . This scale goes to infinity when  $m$  goes to zero, so one must consider non-perturbative solutions [8].

On the other hand, for  $\beta \neq 1$  we have no perturbative vDVZ discontinuity. Indeed, for  $\beta \neq 1$  and  $\beta \neq 1/2$  (so  $\nu \neq 0$  and  $M \neq m$  – see below) we have:

$$a = \frac{\zeta}{r} \left[ e^{-mr} - \frac{1}{4} e^{-Mr} \right], \quad (48)$$

$$b = \frac{\zeta}{m^2 r^3} \left[ (1 + mr) e^{-mr} - (1 + Mr) e^{-Mr} \right] - \frac{3\beta}{4(1 - \beta)} \frac{\zeta}{r} e^{-Mr}, \quad (49)$$

$$c = -\frac{\zeta}{2r} \left[ e^{-mr} + \frac{1}{2} e^{-Mr} \right] - \frac{\zeta}{2m^2 r^3} \left[ (1 + mr) e^{-mr} - (1 + Mr) e^{-Mr} \right], \quad (50)$$

where  $M^2 \equiv m^2(4\beta - 1)/2(1 - \beta)$  is the perturbative mass of the trace  $h_M^M$ , and  $\zeta$  is an integration constant. In the massless limit we have  $a = -r_*/2r$ ,  $b = r_*/2r$  and  $c = \gamma r_*/r$ , where  $r_* \equiv -3\zeta/2$  and  $\gamma = 1/4(1 - \beta)$ , which is the very result we obtained in the beginning of this subsection in constrained gravity.

When  $\beta = 1/2$ , the two masses are degenerate,  $M = m$ , but the above formulas are still valid. We have  $a = -b = -c = \zeta_1 \exp(-mr)/r$ , where  $\zeta_1$  is an integration constant. So for  $\beta \neq 1$  we have no perturbative vDVZ discontinuity.

### 3.3.2 Non-perturbative Asymptotic Solutions for $\beta = 1$

For  $\beta = 1$  the linearized approximation breaks down and we must consider non-perturbative massive solutions. In the massless limit they smoothly go to the asymptotic massless solutions we discussed for  $\beta = 1$  in Subsection 3.2.1. We have:

$$A = 1 - \frac{r_*}{2r} + \mathcal{O}\left(\frac{r_*}{r}\right)^{\frac{3}{2}} + \mathcal{O}(\mu^2 \sqrt{r_* r^3}), \quad (51)$$

$$B = 1 + \sqrt{\frac{8}{39}} \left(\frac{r_*}{r}\right)^{\frac{1}{2}} + \frac{r_*}{2r} + \mathcal{O}\left(\frac{r_*}{r}\right)^{\frac{3}{2}} + \mathcal{O}(\mu^2 \sqrt{r_* r^3}), \quad (52)$$

$$C = r \left[ 1 + \sqrt{\frac{8}{39}} \left(\frac{r_*}{r}\right)^{\frac{1}{2}} + \eta \frac{r_*}{r} + \mathcal{O}\left(\frac{r_*}{r}\right)^{\frac{3}{2}} + \mathcal{O}(\mu^2 \sqrt{r_* r^3}) \right], \quad (53)$$

where  $\eta$  is an integration constant. Note that the expansion in  $\mu^2$  is valid at distance scales  $r \ll 1/\mu$ . As  $\mu \rightarrow 0$ , we have a smooth massless limit for all  $r$ .

### 3.3.3 Comments

Why is all this useful? If  $\beta \neq 1$ , then asymptotic perturbative computations in cases where the conjugate momenta for the relevant degrees of freedom are small (see below) – and this includes static solutions – are valid without invoking the Vainshtein mechanism [8], *i.e.*, there is no *large* scale – such as  $r_2 \sim (r_*/m^2)^{1/3}$  for  $\beta = 1$  – below which the perturbative approximation breaks down. As was argued in [4], while for  $\beta \neq 1$  the trace  $h$  is a ghost, this is a mere artifact of linearization and non-perturbatively the Hamiltonian is bounded from below. Simply put, when



relevant conjugate momenta are large (see [4] for details) – which is precisely when the “ghostliness” of  $h$  would become problematic – the perturbative expansion that produces the fake “ghost”  $h$  is invalid in the first place, and non-perturbatively there is no ghost. So *a priori* there is no reason to discard  $\beta \neq 1$  cases as “bad”. In fact, there is no symmetry that would protect  $\beta$  from quantum corrections. In gravitational Higgs mechanism requiring that  $\beta = 1$  is nothing but a fine-tuning of the vacuum energy density in the unbroken phase against higher-derivative couplings in the scalar sector [11], which fine-tuning is unstable against quantum corrections.

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